Physics of Solitons

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outline

- background for solitons
- examples in solid state physics
- applications to the recent work in cold atom systems
- summary

what is soliton?

Chris Eilbeck & Heriot Watt University 1995

why solitons?

- Linear systems: Maxwell equation, QM, linear response theory, Fourier transform....etc.

(Based on a linear formalism emphasizing superposition principle)

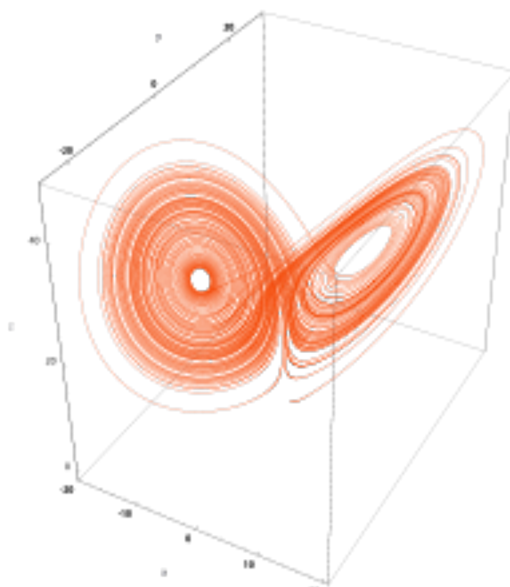
- Non-linear systems: Navier-Stokes equation, collective effects arising from interactions...

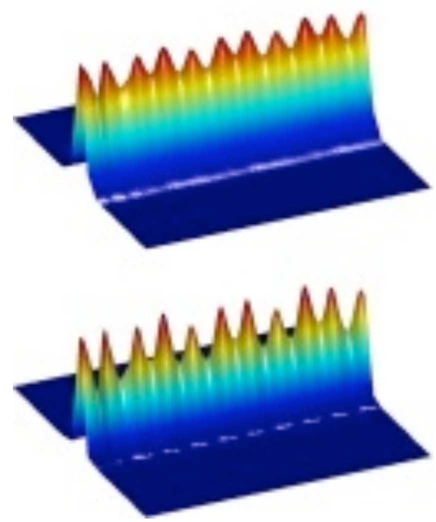
$$\boxed{\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\nabla\vec{v}\right)=-\nabla P+\mu\nabla^{2}\vec{v}+\vec{f}}$$

why solitons?

However, most theoretical approaches lies in linearzing the system and treat the non-linearity as perturbations.

The concept of "intrinsic analysis" of non-linear system lead to the discovery of strange attractor and solitons.





first discovery

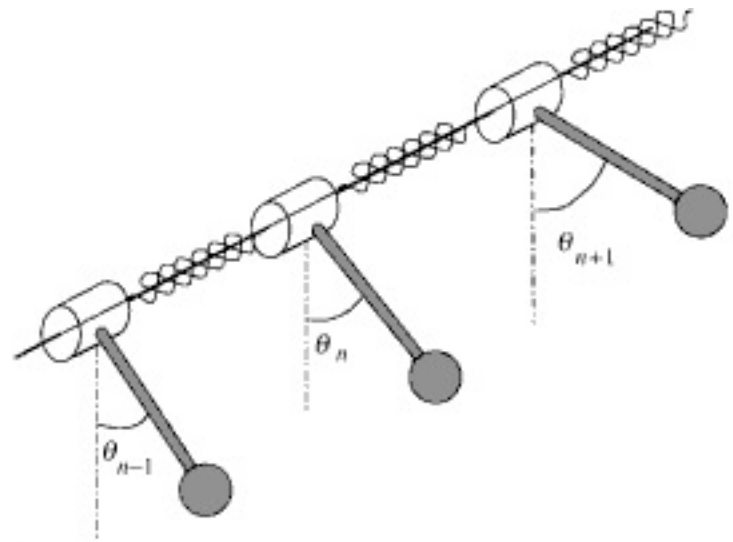
John Scott Russell
(1808~1882)

"I was observing the motion of a boat which was rapidly drawn along the narrow channel.....the boat suddenly stopped- not so the mass of water in the channel.....it accumulated round the pow of vessel.... rolled forward with great velocity, assuming the form of solitary elevation....... apparently without change of form or diminution of speed..."

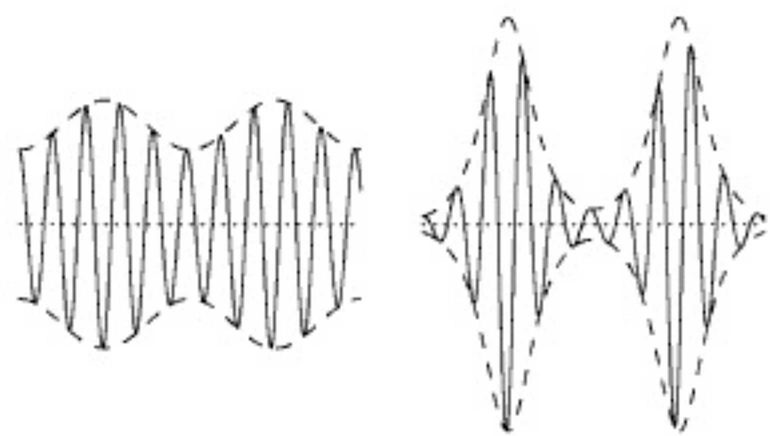
non-linearity?

A simple example is the pendulum chain including the non-linear higher order term. The final result for the amplitude factor in this system leads to the nonlinear Schrodinger equation(NLS).

$$i\frac{\partial\psi}{\partial t} = \left[-P\frac{\partial^2}{\partial x^2} - Q|\psi|^2\right]\psi$$



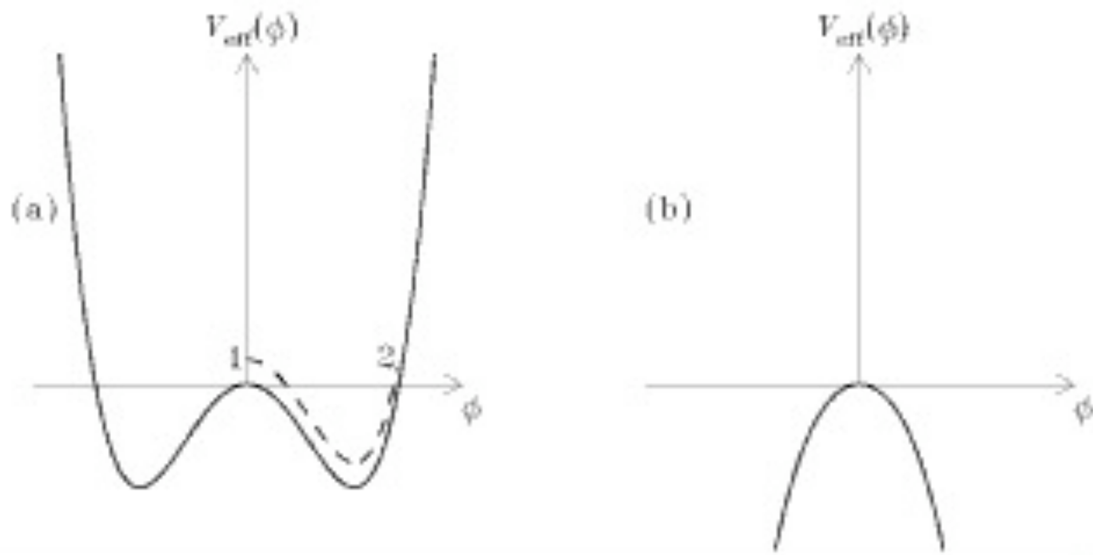
Non-linearity would induce self-modulation into wave packets.



solution of NLS

It can be proved that for systems to have a localized soliton solution, the product PQ must be positive.

$$V_{eff}(\phi)$$



$PQ > 0$ $PQ < 0$

solution of NLS

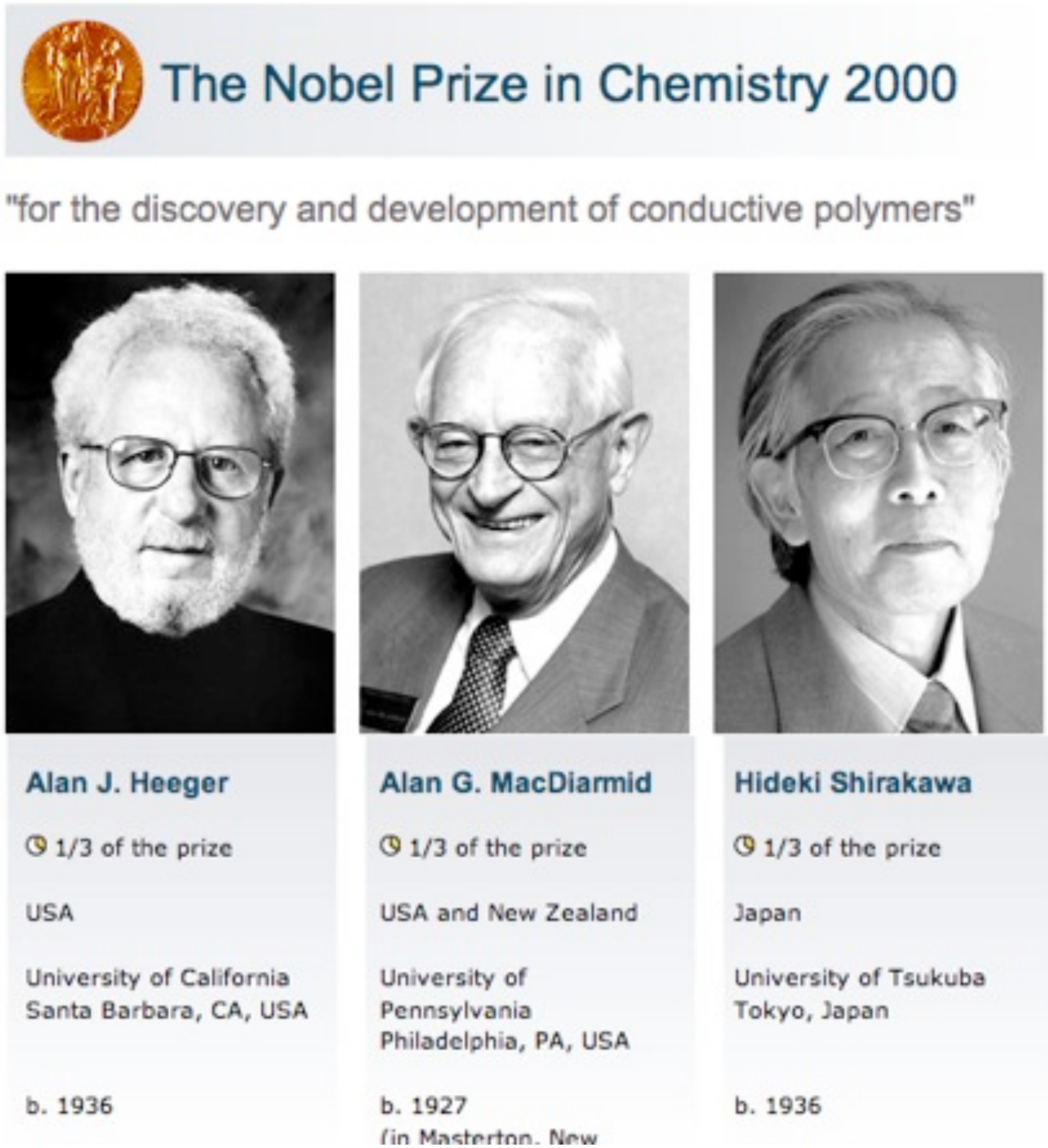
We look for a solution of the form $\psi(x,t) = \phi(x,t)e^{i\theta(x,t)}$ with the carrier wave θ and envelope ϕ have permanent profiles.

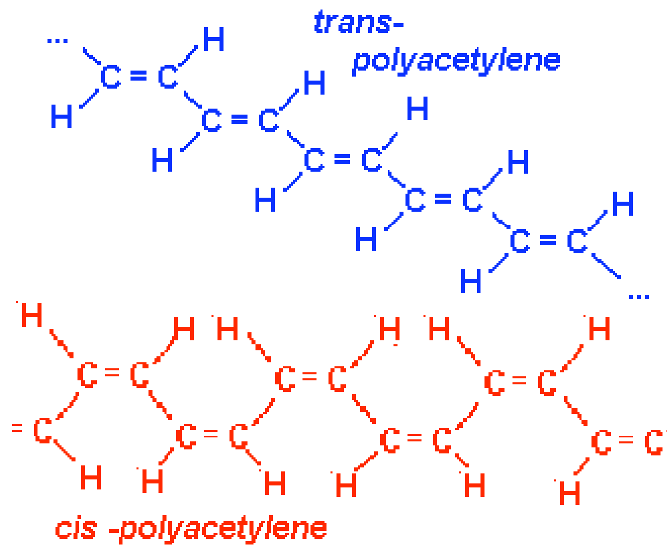
Pseudo-potential argument and localized soliton solution requirement leads to the final solution.

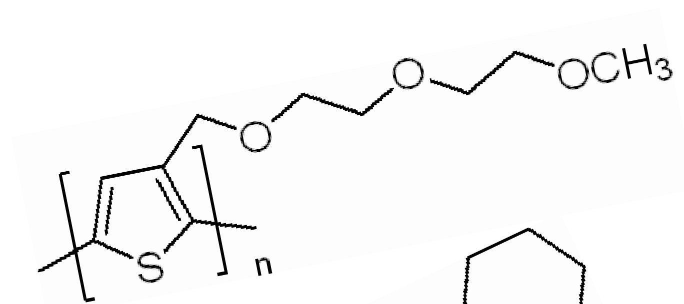
$$\boxed{\psi = \phi_0 \operatorname{sech}\left(\phi_0 \sqrt{\frac{Q}{2P}}(x - u_e t)\right) e^{i\frac{u_e}{2P}(x - u_p t)}}$$

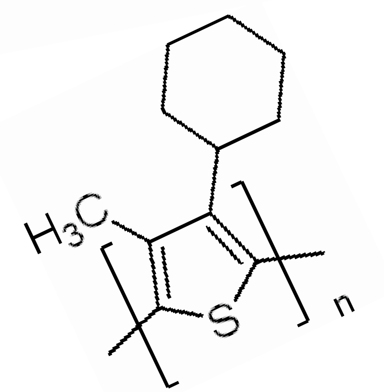
in Solid State Physics?

solitons in conducting polymers







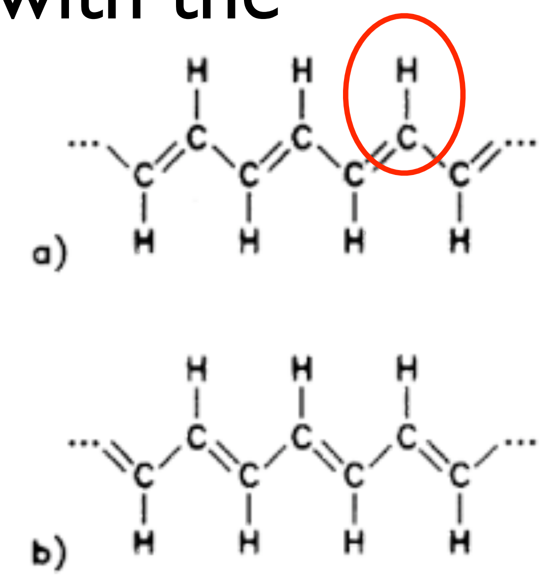


solitons in polyacetylene

the static model proposed in PRB in 1980

- Pi-electrons: the switching of pi electrons from one bond to another leads to the two states A and B.

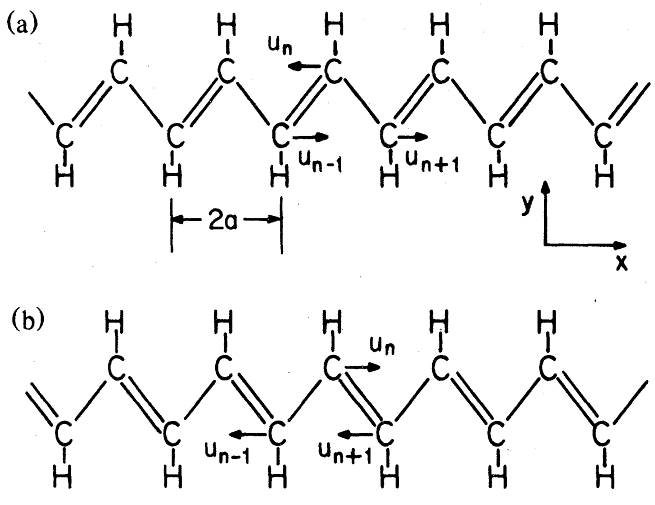
- distortions of carbon chains: due to the bond length difference between sigma and pi bonds, the switching of pi electrons is coupled with the distortions of the lattice.



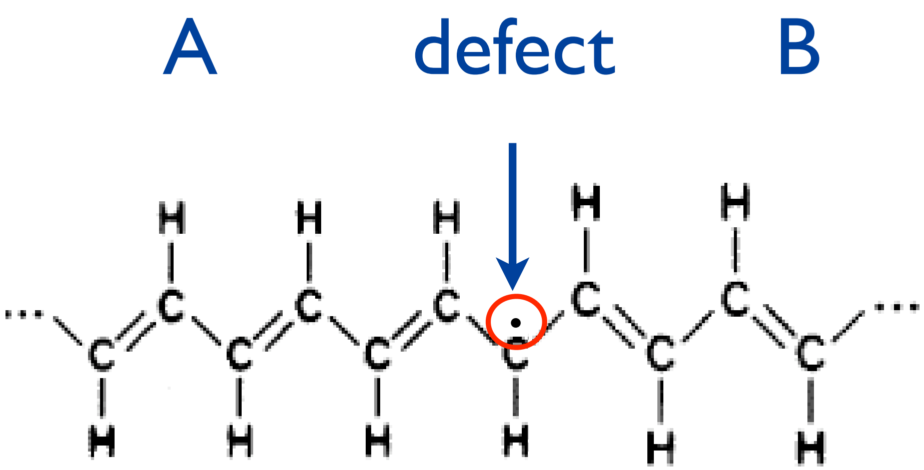
$$\mathcal{H} = \mathcal{H}_\sigma + \mathcal{H}_\pi$$

solitons in polyacetylene

trans

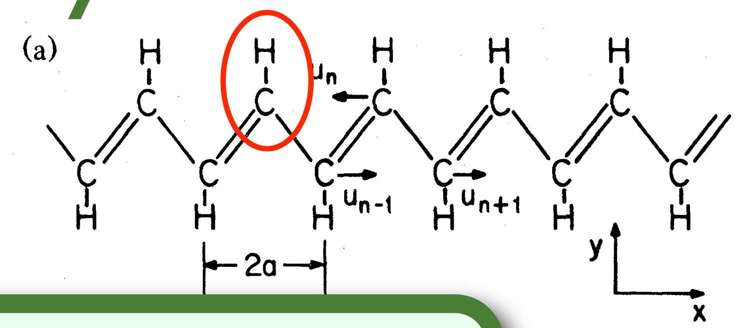


these two *trans*-polyactelene types are identical in geometry thus are energetically degenerate



if they coexist in one molecule, defects, which is topological in nature, would appear in the connection between the two types.(topological soliton)

solitons in polyacetylene



Hamiltonian in dimerised picture

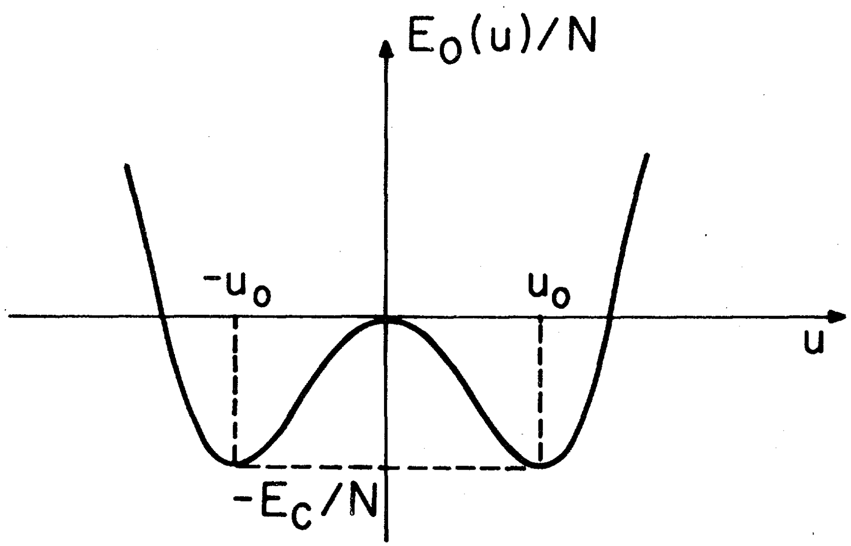
$$\begin{aligned}\mathcal{H} &= \mathcal{H}_\sigma + \mathcal{H}_\pi \\ &= \sum_n \frac{p_n^2}{2m} + \frac{1}{2}K(u_n - u_{n+1})^2 \\ &- \sum_n [t_0 - \alpha(u_{n+1} - u_n)](c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1})\end{aligned}$$

this tight-bonding description takes into account the non-linearity between contributions from two bonds.

band structure

assume the regular distortion $u_n = (-1)^n u$, using the standard band theory calculation:

$$E_0(u) = 2KNu^2 - 2\int_{-\pi/2a}^{\pi/2a} dk \frac{Na}{2\pi}\sqrt{4t_0^2\cos^2 ka + 16\alpha^2 u^2 \sin^2 ka}$$



the band structure implies the validity of the dimerised picture for the ground state of polyacetylene.

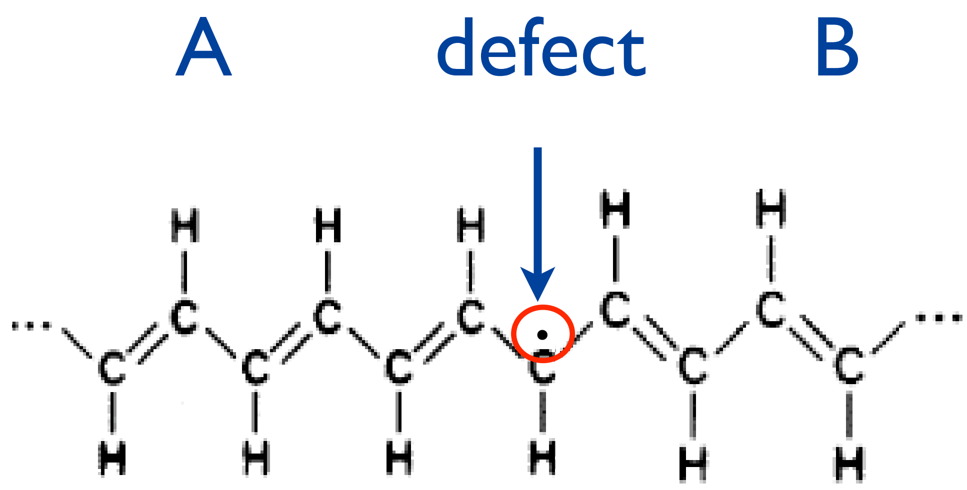
excited state(soliton)

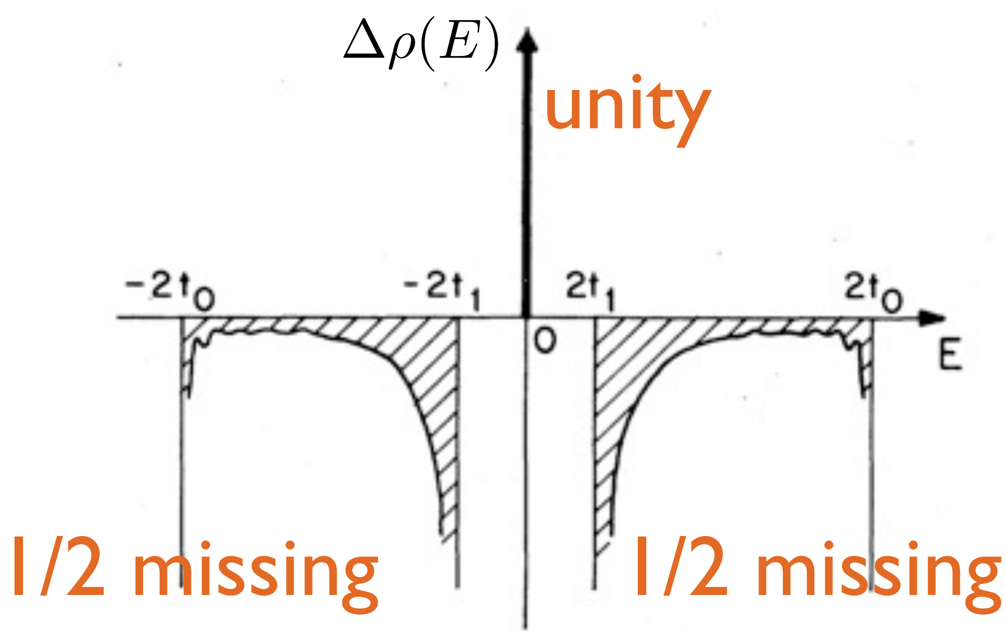
The study of the ground of the dimerised chain is twofold degenerate. Associated with this degenercy, we expect there to exit an *elementary excitation* corresponding to a soliton.

$$\mathcal{H}' = A\int_{-\infty}^{\infty} d\xi \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial\xi}\right)^2 + \frac{1}{4}(1-\phi^2)^2\right]$$

performing canonical transformation, we obtain a PDE having a soliton-like solution. Also, the soliton energy can be calculated by plugging the experimentally observed data(lies in the middle of bandgap).

conduction mechanism



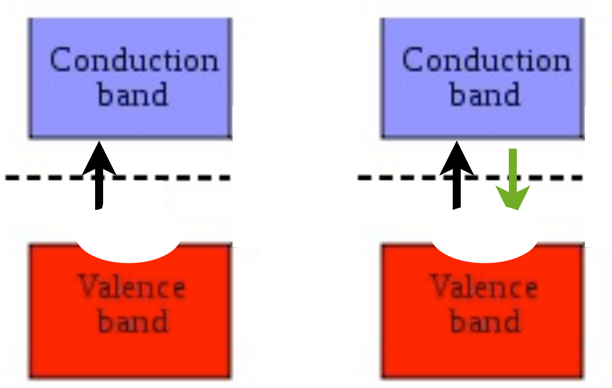


The distortion of the CH lattice is associated with the appearance of an electronic energy state which is localized around the soliton center.

Can be visualized by the change in the density of state due to the presence of soliton.

conduction mechanism

It could be calculated that the energy level associated with soliton is situated between CB and VB and is occupied by the defect (PI)electron. Thus neutral soliton carries no charge with spin 1/2(unpaired.)



neutral soliton charged soliton

By doping with a electron doner, the soliton can be viewd as a pseudo-particle with charge (-e) and S=0

Solitons in BEC

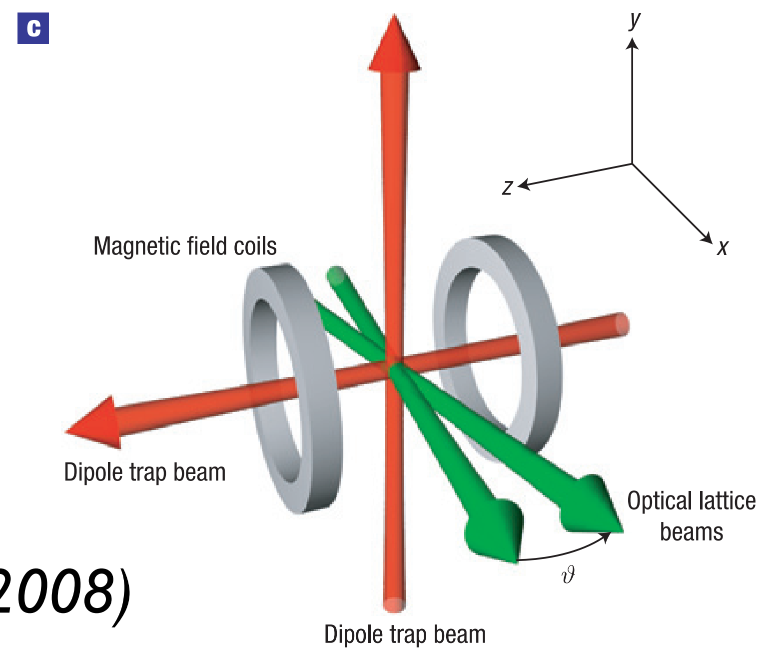
BEC stability

Homogeneous BEC with purely contact interaction:

- repulsive (a>0): stable
- attractive(a<0): unstable

Being long range and anisotropic, the dipole-dipole interaction (DDI) changes the stability condition of the system.

$$V_{trap} = \frac{1}{2} m(\omega_r^2 r^2 + \omega_z^2 z^2)$$



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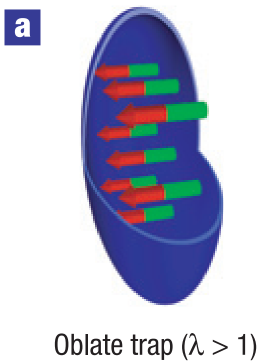
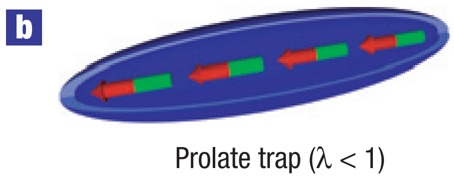
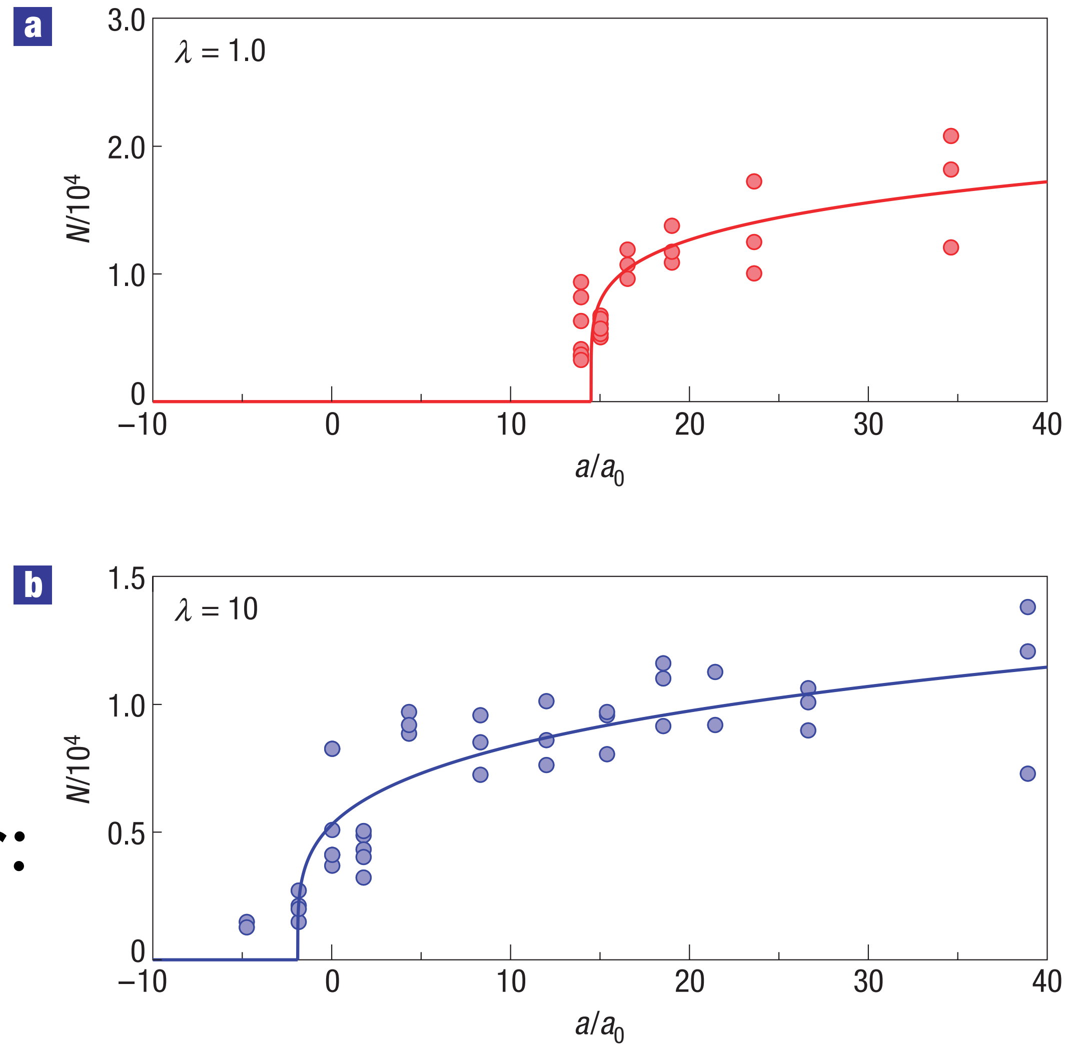
BEC stability

pancake





cigar



(a) dipoles repel each other:
BEC stable

(b) dipoles attract each other:
dipolar collapse

solitons in BEC

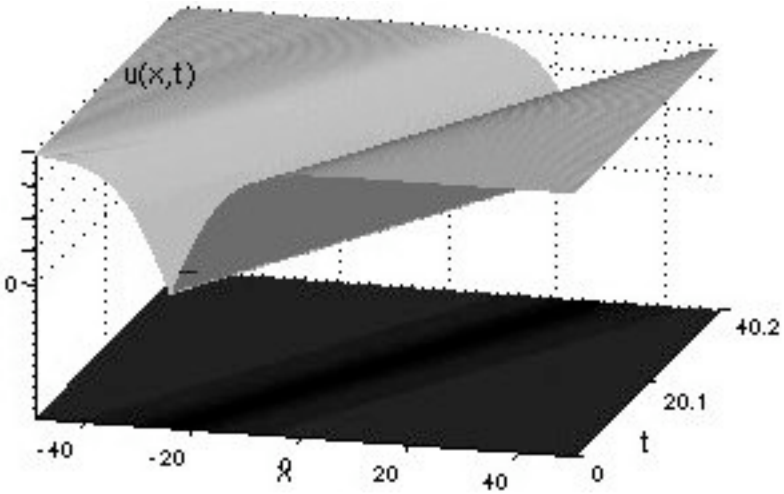
Consider a 3D cigar shape condensate confined in a trap with aspect ration <<1. We can map the GP equation into a 1D effective one which has the form of NLSE.

$$\boxed{\left[i\frac{\partial}{\partial\tau}+\frac{1}{2}\frac{\partial^2}{\partial s^2}+Q|\psi(s,\tau)|^2\right]\psi(s,\tau)=\frac{1}{2}\lambda^2 s^2\psi(s,\tau)}$$

This equation admits a soliton-like solution, however, for larger dimension, NLSE does *not* allow for such a solution.

solitons in BEC

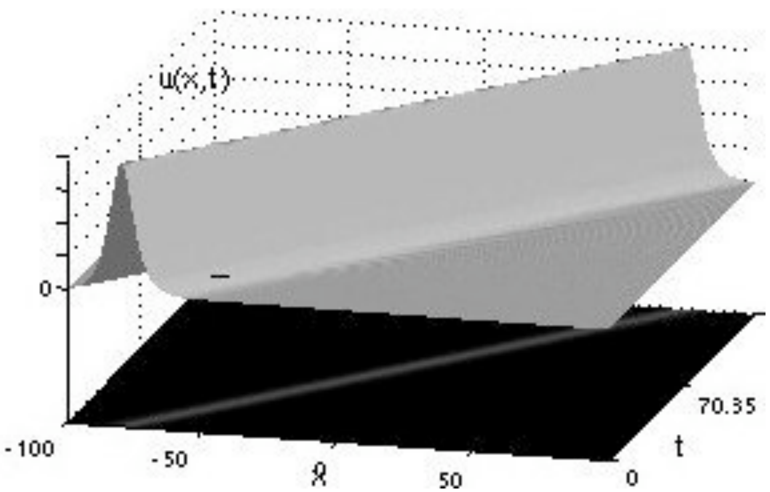
Repulsive interaction(a>0): dark or grey soliton



$$s = z/l$$

$$\boxed{\psi(s,\tau) = A_0 \tanh(A_0\sqrt{2}s)e^{-2iA_0^2\tau}}$$

Attractive interaction(a<0): bright soliton



$$\boxed{\psi(s,\tau) = \frac{N}{l^2}\sqrt{\frac{-a}{2\pi}}\operatorname{sech}\left(\frac{aNs}{l}\right) e^{-\rho^2/2}e^{-iv_\rho\tau}}$$

phonon instability

At sufficiently low temperature, the physics of dipolar BEC can be described by a nonlinear Schrodinger equation.

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + g|\Psi(\mathbf{r},t)|^2 + \int d\mathbf{r}' V_d(\mathbf{r}-\mathbf{r}')|\Psi(\mathbf{r}',t)|^2\right]\Psi(\mathbf{r},t)$$

Also assume that all dipoles are oriented along the axis so that DDI: $V_d(\mathbf{R}) = (d^2/R^3)(1-3\cos^2\theta)$.It can be shown that in this 3D case, there exist low momentum excitation, called phonon instablity(PI), leading to condensate collapse.

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PI in 2D?

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Phonon Instability with Respect to Soliton Formation in Two-Dimensional Dipolar Bose-Einstein Condensates

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The partially attractive character of the dipole-dipole interaction leads to phonon instability in dipolar Bose-Einstein condensates, which is followed by collapse in 3D geometries. We show that in 2D, the nature of the post-instability dynamics is fundamentally different, due to the stabilization of 2D solitons. As a result, a transient gas of attractive solitons is formed, and collapse may be avoided. In the presence of an harmonic trap, the post-instability dynamics is characterized by a transient pattern formation followed by the creation of stable 2D solitons. This dynamics should be observable in ongoing experiments, allowing for the creation of stable 2D solitons for the first time ever in quantum gases.

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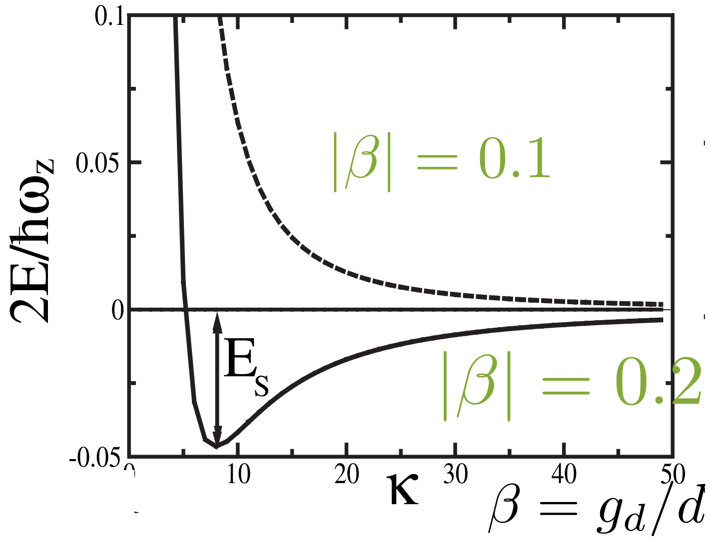
Unlike in 3D, 2D PI does *not necessarily* lead to condensate collapse. Instead, the absence is explained by the *formation of 2D bright solitons*.

solitons in 2D?

But does there exist soliton-like solution in 2D in the presence of DDI?



$$\Psi_0(\mathbf{r}) = \frac{1}{\pi^{3/4} l_z^{3/2} L_\rho L_z^{1/2}} \exp\left(-\frac{x^2+y^2}{2l_z^2 L_\rho^2} - \frac{z^2}{2l_z^2 L_z^2}\right)$$

As proposed in literature, 2D solitary wave is stabilized by the DDI:

$$|\beta| > 3/8\pi$$

why PI in 2D is stable?

Consider the homogeneous dipolar BEC in the x-y plane, with a strong harmonic confinement in the z direction. The system can be considered "frozen" into the ground state $\phi_0(z)$, thus the factorization of the wave function:

$$\Psi(\vec{r}) = \psi(\vec{\rho})\phi_0(\vec{z})$$

Factorization+Convolution theorem+ Fourier transform +Bogoliubov equation leads to the final dispersion:

$$\boxed{\epsilon(\mathbf{k})^2 = E_k^2 + \frac{2gn_0E_k}{\sqrt{2\pi}l_z}\left[1 + \frac{4\pi\beta}{3}h_{2D}\left(\frac{kl_z}{\sqrt{2}}\right)\right]}$$

orientation of dipoles

In the 2D case, the orientation of dipole plays an important role.

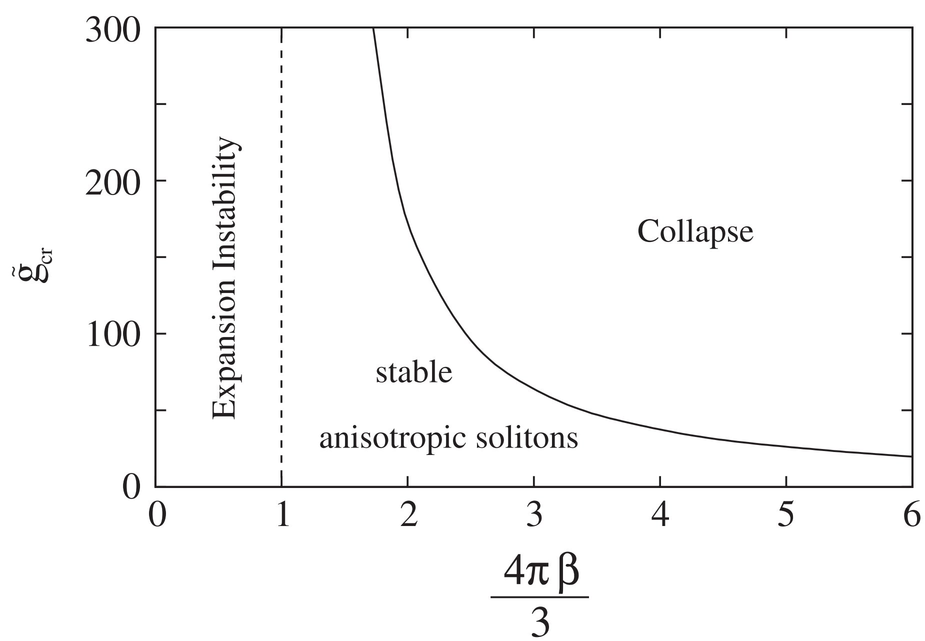
$$h_{2D,\parallel}(\vec{\kappa}) = -1 + 3\sqrt{\pi/2}(\kappa_x^2/k)e^{\kappa^2}\mathrm{erfc}(\kappa)$$

$$h_{2D,\perp}(\vec{\kappa}) = 2 - 3\sqrt{\pi}e^{\kappa^2}\mathrm{erfc}(\kappa)$$

In the perp($\perp$) configuration, stable bright solitons are possible for g>0 if $\beta < -3/8\pi$, which is the same condition for the formation of 2D soliton. This stable soliton can be shown to be *isotropic*.

orientation of dipoles

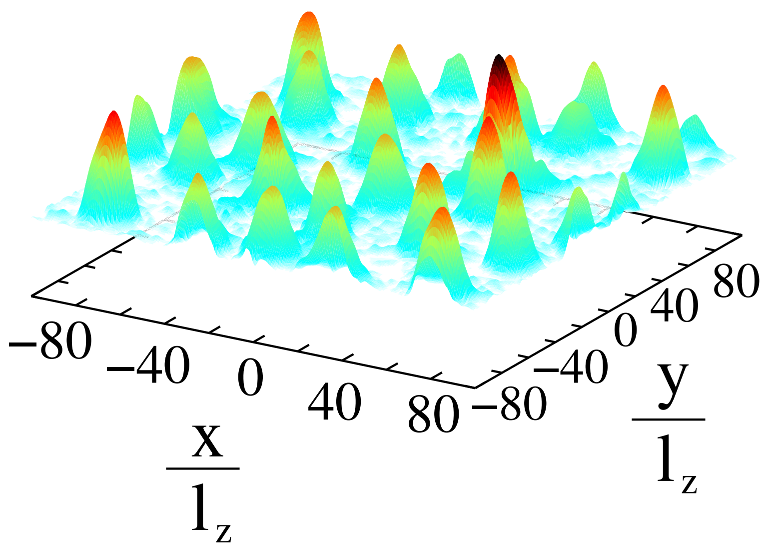
In the parallel(∥) configuration, the calculated stable condition is: $\beta > 3/4\pi$. However, another constraint should be imposed.



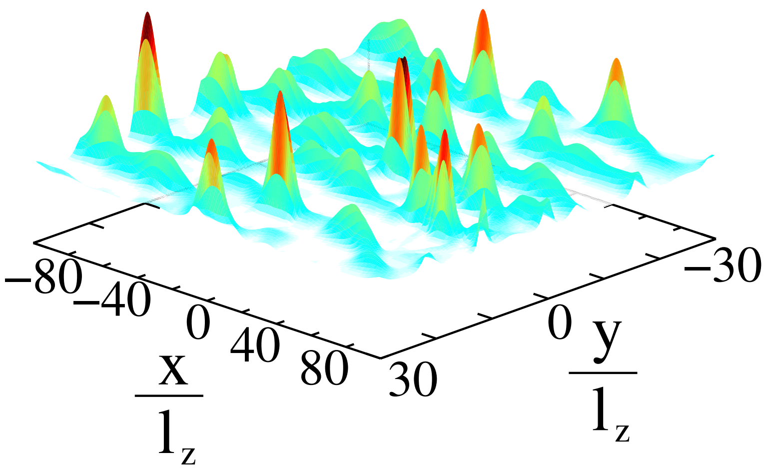
There is a critical value $\tilde{g}_c(\beta)$ such that for N>Nc, the minimum of the energy functional disappears.

$$\boxed{\tilde{g}_c(\beta) \equiv gN_c/\sqrt{2\pi}l_z}$$

formation of 2D solitons



the perp($\perp$) configuration



the parallel($\parallel$) configuration

summary

- In the perp configuration, the isotropic solitons are stable as long as the system remains effectively in 2D and $\beta < -3/8\pi$
- In the parallel configuration, the anisotropic soliton may collapse when surpassing a critical number of particles per soliton even $\beta > 3/4\pi$

comment

- Does physical soliton exist?

  a physical system is never described by an equation having true soliton solution

- Soliton is everywhere?
- A debt concerning solitons in biological molecules

Thank you!!