PROOF OF ORE'S THEOREM

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1. Ore's Theorem

In this note, we prove the following result:

Theorem 1. *Suppose that G is a simple graph with n vertices, $n \geq 3$ and $deg(x) + deg(y) \geq n$ whenever x and y are non-adjacent vertices in G, then G has a Hamilton circuit.*

Proof. We prove by contradiction:

(a) Suppose that $\deg(x) + \deg(y) \geq n$ for every pair of nonadjacent vertices x and y in G. If G does not have a Hamilton circuit, continue as long as possible in adding missing edges to G(since G is not necessarily a complete graph) one at a time in such a way that we do not obtain a graph with a Hamilton circuit. This cannot go on forever, because once we've formed the complete graph by adding all missing edges, there's a Hamilton circuit. Whenever the process stops, we have obtained a (necessarily non-complete) graph H with the desired property.

(b) If we add one more edge to H, this produce a Hamilton circuit, which uses all added edges. The path consisting of this circuit with the added edge omitted is the Hamilton path in H.

(c) Let $v_1, v_2, \cdots v_n$ be a Hamilton path in H. Clearly v_1 and v_n are not adjacent in H, because H has no Hamilton circuit. Therefore they are not adjacent in G. By hypothesis,

$$\deg(v_1) + \deg(v_n) \geq n$$

We rewrite in the other form,

$$(n-1) - \deg(v_n) \leq \deg(v_1) - 1$$

, the left-hand-side(LHS) is just the number of vertices not adjacent to v_n(not include v_n itself).

(d) Let S be the set of vertices preceding each vertex adjacent to v_1 in the Hamilton path. Because there's no vertex following v_n, so $v_n \notin S$. Each one of the $\deg(v_1)$ vertices adjacent to v_1 gives rise to one element in S. So $|S| = \deg(v_1)$

(e) By (c), there are at most $\deg(v_1) - 1$ vertices other than v_n not adjacent to v_n, and by (d) there are $\deg(v_1)$ vertices in S, none of which is v_n. Therefore at least one vertex of S is adjacent to v_n(since each of the $\deg(v_1)$ vertices in S is either *adjacent* or *not adjacent* to v_n, but there are *at most* $\deg(v_1) - 1$ vertices

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not adjacent to v_n). Let v_k be such an vertex and, by definition, H contains the edges $\{v_k v_n\}$, $\{v_{k+1} v_1\}$, where $1 < k < n_1$.

(f) Now $v_1, v_2, \cdots v_{k-1}, v_k, v_n, v_{n-1} \cdots, v_{k+1}, v_1$ is a Hamilton circuit in H, which contradicts our construction of H. Therefore our assumption that G did not originally have a Hamilton circuit is wrong, and our proof by contradiction is complete. □

References

[1] K. H. Rosen *Discrete Mathematics and its applications, 6th ed.*, McGraw-Hill, 2007
[2] R. P. Grimaldi, *Discrete and Combinatorial Mathematics, 5th ed.*, Addison Wesley, 2004.

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