THE MARRIAGE PROBLEM.*

By Paul R. Halmos and Herbert E. Vaughan.

In a recent issue of this journal Weyl[1] proved a combinatorial lemma which was apparently considered first by P. Hall.[2] Subsequently Everett and Whaples[3] published another proof and a generalization of the same lemma. Their proof of the generalization appears to duplicate the usual proof of Tychonoff's theorem.[4] The purpose of this note is to simplify the presentation by employing the statement rather than the proof of that result. At the same time we present a somewhat simpler proof of the original Hall lemma.

Suppose that each of a (possibly infinite) set of boys is acquainted with a finite set of girls. Under what conditions is it possible for each boy to marry one of his acquaintances? It is clearly necessary that every finite set of k boys be, collectively, acquainted with at least k girls; the Everett-Whaples result is that this condition is also sufficient.

We treat first the case (considered by Hall) in which the number of boys is finite, say n, and proceed by induction. For $n = 1$ the result is trivial. If $n > 1$ and if it happens that every set of k boys, $1 \leqq k < n$, has at least $k + 1$ acquaintances, then an arbitrary one of the boys may marry any one of his acquaintances and refer the others to the induction hypothesis. If, on the other hand, some group of k boys, $1 \leqq k < n$, has exactly k acquaintances, then this set of k may be married off by induction and, we assert, the remaining $n - k$ boys satisfy the necessary condition with respect to the as yet unmarried girls. Indeed if $1 \leqq h \leqq n - k$, and if some set of h bachelors were to know fewer than h spinsters, then this set of h bachelors together with the k married men would have known fewer than $k + h$ girls. An

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[1] H. Weyl, "Almost periodic invariant vector sets in a metric vector space," *American Journal of Mathematics*, vol. 71 (1949), pp. 178-205.

[2] P. Hall, "On representation of subsets," *Journal of the London Mathematical Society*, vol. 10 (1935), pp. 26-30.

[3] C. J. Everett and G. Whaples, "Representations of sequences of sets," *American Journal of Mathematics*, vol. 71 (1949), pp. 287-293. Cf. also M. Hall, "Distinct representatives of subsets," *Bulletin of the American Mathematical Society*, vol. 54 (1948), pp. 922-926.

[4] C. Chevalley and O. Frink, Jr., "Bicompactness of Cartesian products," *Bulletin of the American Mathematical Society*, vol. 47 (1941), pp. 612-614.

application of the induction hypothesis to the $n-k$ bachelors concludes the proof in the finite case.

If the set B of boys is infinite, consider for each b in B the set $G(b)$ of his acquaintances, topologized by the discrete topology, so that $G(b)$ is a compact Hausdorff space. Write G for the topological Cartesian product of all $G(b)$; by Tychonoff's theorem G is compact. If $\{b_1, \cdots, b_n\}$ is any finite set of boys, consider the set H of all those elements $g = g(b)$ of G for which $g(b_i) \neq g(b_j)$ whenever $b_i \neq b_j$, $i, j = 1, \cdots, n$. The set H is a closed subset of G and, by the result for the finite case, H is not empty. Since a finite union of finite sets is finite, it follows that the class of all sets such as H has the finite intersection property and, consequently, has a non empty intersection. Since an element $g = g(b)$ in this intersection is such that $g(b') \neq g(b'')$ whenever $b' \neq b''$, the proof is complete.

It is perhaps worth remarking that this theorem furnishes the solution of the celebrated problem of the monks.[5] Without entering into the history of this well-known problem, we state it and its solution in the language of the preceding discussion. A necessary and sufficient condition that each boy b may establish a harem consisting of $n(b)$ of his acqaintances, $n(b) = 1, 2, 3, \cdots$, is that, for every finite subset B_0 of B, the total number of acquaintances of the members of B_0 be at least equal to $\Sigma n(b)$, where the summation runs over every b in B_0. The proof of this seemingly more general assertion may be based on the device of replacing each b in B by $n(b)$ replicas seeking conventional marriages, with the understanding that each replica of b is acquainted with exactly the same girls as b. Since the stated restriction on the function n implies that the replicas satisfy the Hall condition, an application of the Everett-Whaples theorem yields the desired result.

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[5] H. Balzac, *Les Cent Contes Drôlatiques*, IV, 9: *Des moines et novices*, Paris (1849).